B. Math. Hons. Backpaper Examination
Algebra - I
Ist semester 200-2001

Solve any six questions - — - The test is of 2 hours duration.

1. Prove that every matrix $g \in GL_2(\mathbf{C})$ is a product of elementary matrices.
2. Prove Cayley's theorem : any finite group is isomorphic to a subgroup of a permutation group S_n.
3. (i) Find subgroups $H \subseteq K \subseteq G$ such that H is normal in K, K is normal in G but H is not normal in G.

   (ii) Prove that any normal subgroup of a group G arises as the kernel of a homomorphism on G.
4. Let H be a subgroup of a finite group G. Show that the number of subgroups of G which are of the form $gHg^{-1}; g \in G$ is $[G : N_G(H)]$ where $N_G(H) \stackrel{d}{=} \{g \in G : gHg^{-1} = H]$.
5. Let F be an infinite field and V be a finite dimensional vector space over F. Show that $V \neq \bigcup\limits_{i=1}^{n} V_i$ for proper subspaces $V_1, \cdots, V_n$ of V.
6. If $\{v_1, \ldots, v_n\}$ is a basis of a vextor space V over some field F, show that any basis of V must be of the form $(v_1 \ldots v_n)g$ for some $g \in GL_n(F)$.
7. If V_1, V_2 are subspaces of a finite-dimensional vector space V over a field F, prove that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

8. Let F be a field and $A \in M_n(F)$. Regarding F^n as the vector space of n-tuples written as column vectors, prove that the set $\{X \in F^n : AX = 0\}$ is a subspace of dimension equal to $n - \dim W$ where $W \subseteq F^n$ is the subspace generated by columns of A.